

Fourrier transform of an impulsion train

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This document was inspired by http://math.ut.ee/~toomas_1/harmonic_analysis/ and gives the demonstration of the following theorem :

Theorem 1 (Impulsion train). *The Fourier transform of a spatial domain impulsion train of period T is a frequency domain impulsion train of frequency $\Omega = 2\pi/T$.*

$$\boxed{\sum_{p \in \mathbb{Z}} \delta(x - pT) \xleftrightarrow{FT} \Omega \sum_{k \in \mathbb{Z}} \delta(x - k\Omega)} \quad (1)$$

Reminders

Fourier Coefficients

Let f be a T -periodic function, we have :

$$f(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\Omega x} \quad \text{with} \quad \begin{cases} \Omega = \frac{2\pi}{T} \\ c_k = \frac{1}{T} \int_0^T f(t) e^{-ik\Omega t} dt \end{cases}$$

The c_k are called *the Fourier coefficients* of f . This coefficient can be rewritten as an integral over any interval of length T . In particular, we will use :

$$\boxed{c_k = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-ik\Omega t} dt} \quad (2)$$

Proof. Let $g(t) = f(t)e^{-ik\Omega t}$. It is a T -periodic function since we have :

$$\begin{aligned} g(t+T) &= f(t+T)e^{-ik\Omega(t+T)} = f(t)e^{-ik\Omega t} e^{-ik\Omega T} \\ &= f(t)e^{-ik\Omega t} \underbrace{e^{-ik2\pi}}_{=1} = g(t) \end{aligned}$$

Equation (2) is said due to the fact that the integral of a T -periodic function is constant over any interval of length T as can be seen from :

$$\begin{aligned} \int_c^{c+T} g(t)dt &= \int_c^0 g(t)dt + \int_0^T g(t)dt + \underbrace{\int_0^c g(t)dt}_{t=u+T} \\ &= \int_c^0 g(t)dt + \int_0^T g(t)dt + \int_0^c g(u)du = \int_0^T g(t)dt \end{aligned}$$

□

Fourier Transform

The *Fourier transform* $F(\omega)$ of a real-valued function $f(x)$ is defined by :

$$F(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x} dx \quad (3)$$

The *inverse Fourier transform* is given by the relation

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)e^{i\omega x} d\omega \quad (4)$$

When two functions are related by the Fourier transform, we note :

$$f(x) \xleftrightarrow{FT} F(\omega)$$

We have the *symmetry property* :

$$\boxed{\text{if } f(x) \xleftrightarrow{FT} F(\omega) \text{ then } F(x) \xleftrightarrow{FT} 2\pi f(-\omega)} \quad (5)$$

and the *linearity property* :

$$\boxed{\text{if } \begin{cases} f(x) \xleftrightarrow{FT} F(\omega) \\ g(x) \xleftrightarrow{FT} G(\omega) \end{cases} \text{ then } (\lambda f + g)(x) \xleftrightarrow{FT} (\lambda F + G)(\omega)} \quad (6)$$

The Dirac impulsion

The Dirac function $\delta(x)$ has the *sifting* property. If f is continuous at point a :

$$\boxed{\int_{-\infty}^{\infty} f(t)\delta(t-a)dt = f(a)} \quad (7)$$

The Fourier transform of a translated Dirac is a complex exponential :

$$\boxed{\delta(x - a) \xleftrightarrow{FT} e^{-ia\omega}} \quad (8)$$

Impulsion train

Let's consider $it(x) = \sum_{p \in \mathbb{Z}} \delta(x - pT)$ a train of T -spaced impulsions and let's compute its Fourier transform. We first rewrite f using its Fourier coefficients :

$$it(x) = \sum_{k \in \mathbb{Z}} c_k e^{ik\Omega x}$$

where $\Omega = 2\pi/T$. Using Eq. (2), we have :

$$\begin{aligned} c_k &= \frac{1}{T} \int_{-T/2}^{T/2} it(t) e^{-ik\Omega t} dt = \frac{1}{T} \int_{-T/2}^{T/2} \sum_{p \in \mathbb{Z}} \delta(t - pT) e^{-ik\Omega t} dt \\ &= \frac{1}{T} \sum_{p \in \mathbb{Z}} \int_{-T/2}^{T/2} \delta(t - pT) e^{-ik\Omega t} dt \end{aligned}$$

Since the function $t \mapsto \delta(t - pT)$ is null over the interval $[-T/2, T/2]$ for $p \neq 0$, we are left with only one term in the summation :

$$c_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-ik\Omega t} dt = \frac{1}{T} \underbrace{\int_{-\infty}^{\infty} \delta(t) e^{-ik\Omega t} dt}_{= e^{-i\Omega 0} = 1 \text{ by (7)}} = \frac{1}{T}$$

So finally we have an expression of the impulse train :

$$\boxed{it(x) = \frac{1}{T} \sum_{k \in \mathbb{Z}} e^{ik\Omega x}} \quad (9)$$

Applying the symmetry property (5) to the Fourier transform of a Dirac (8) we find :

$$\begin{aligned} e^{-ik\Omega(-x)} &\xleftrightarrow{FT} 2\pi\delta(-(-x) - k\Omega) \\ e^{ik\Omega x} &\xleftrightarrow{FT} 2\pi\delta(x - k\Omega) \end{aligned}$$

Applying linearity (6) to expression (9) we finally get the equality (1). □